Quantum Hall wave functions on the torus

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(Received 4 December 2007; published 18 March 2008)

We present explicit expressions for a large set of hierarchy wave functions on the torus. Included are the Laughlin states, the states in the positive Jain series, and recently observed states at, e.g., $\nu=4/11$. The techniques we use constitute a nontrivial extension of the conformal field theory methods developed earlier to construct the corresponding wave functions in disk geometry.

DOI: 10.1103/PhysRevB.77.125321 PACS number(s): 73.43.Cd, 71.10.Pm, 11.25.Hf

I. INTRODUCTION

A basic concept in the physics of the fractional quantum Hall effect is that of an incompressible liquid. Theoretically, there are two main types of such liquids depending on the quantum statistics of the quasiparticles: the Abelian and the non-Abelian liquids. Of these, only the former are firmly established experimentally. The Abelian liquids are further divided into single- and multicomponent ones, where the former include the prominent filling fractions corresponding to the Laughlin states at $\nu=1/(2q+1)$ (Ref. 1) and the composite fermion states in the Jain series $\nu=n/(2qn\pm1)$.[2] Multicomponent liquids can form in systems with spin or pseudospin degrees of freedom, i.e., in partially polarized or bilayer systems.[3]

There has been two major approaches to the Abelian single-component quantum Hall (QH) liquids, the Haldane–Halperin hierarchy[4,5] and the composite fermion (CF) approach of Jain.[2,6] In the former scheme, the quasiparticles that exist in the vicinity of a given Laughlin state condense into a new Laughlin-like state; this procedure can be repeated, thus forming an ever more complicated hierarchy of coexisting incompressible liquids, while in the CF scheme the prominent fractions are understood in terms of filled effective CF Landau levels. A major advantage of the CF approach is that it provides explicit, and numerically very accurate, wave functions, while an appealing feature of the hierarchy scheme is that it treats all filling fractions on the same footing, including the newly observed ones[7] at, e.g., $\nu=4/11$.

In a series of recent papers,[8–12] we have given a concrete realization of the Haldane–Halperin hierarchy both by providing a large set of explicit and testable wave functions derived using conformal field theory (CFT) techniques and by constructing an exactly solvable model that describes interacting electrons in the lowest Landau level on a thin torus. The CFT construction yields candidate wave functions for every filling fraction obtained by successive condensations of quasielectrons (as opposed to quasiholes) that reduce to the exact solutions on the thin torus and coincide with those of Laughlin and Jain, whenever these exist. The Laughlin states are the exact ground states of a short range model Hamiltonian, but no such Hamiltonian is known for any other hierarchical state. Both the Laughlin and Jain wave functions are, however, known to be excellent approximations to the numerically found Coulomb ground states for small systems.

The standard way to quantitatively decide if a proposed QH wave function captures the physics of a given phase is to study small systems numerically, typically by exact diagonalization, and compare the numerically obtained energies and wave functions with the proposed ones. To carry out such comparisons, one has to choose a geometry. Early calculations were performed in a physically motivated disk geometry, but rather than using an explicit confining potential, the total angular momentum was fixed. The drawback of this method is that small systems might exhibit large, and unphysical, edge effects. These can be eliminated by using a finite geometry, and most large scale calculations have been performed on the sphere. Wave functions on the torus, on the other hand, turn out to be more complicated, but this geometry provides other advantages. Not only are there no edges, but there is also no curvature, and since the torus is a genus one manifold, the topological properties of the different phases will be reflected in the ground state degeneracy.[13] Also, on a torus, the one-dimensional nature of a Landau level is explicit: There is a natural mapping of the two-dimensional problem onto a one-dimensional lattice model that is solvable on the thin torus. There has been considerable progress in understanding various phases of the QH system in terms of exactly solvable models that describe the thin torus,[8,11,14–18] (for details, see Ref. 19).

Altogether, we think that there are strong reasons to construct the hierarchy wave functions on the torus: It will facilitate comparisons with numerical calculations, it will give information about the topological order via the ground state degeneracies, and it will provide a way to extend the exact results in the solvable thin torus limit to the experimentally accessible regime corresponding to a thick torus. Explicit wave functions on the torus have previously only been obtained in a few special cases where the structure is simple.[20,21] In particular, the torus version of Jain or the hierarchy wave functions have, to our knowledge, not been constructed.

In the present paper, we extend the CFT construction to obtain explicit wave functions in the torus geometry. In Sec. II, we briefly review the necessary CFT machinery, as well as some technical details about the lowest Landau level on a torus. In Sec. III, we use this to construct wave functions on the torus. We analyze the Laughlin states (one field) and the $\nu=2/5$ state (two fields) in some detail, and then provide the

basic relations and logical steps needed to derive the wave function for a general hierarchy state that is obtained from a Laughlin state by successive condensations of quasielectrons. In Sec. V, we describe alternative charge vectors that simplify the structure of the wave functions and are very useful for explicit calculations but not suitable for deriving the general formulas of the previous section. In Sec. VI, we compare the $\nu=2/5$ wave function with the result of an exact numerical diagonalization by calculating overlaps. Section VII includes some concluding remarks and an outlook. Several technical steps are explained in Appendixes A and B, and as a service to the practically oriented reader, we also give explicit expressions for the states at $\nu=3/7$ and $\nu=4/11$ in Appendix C.

II. HIERARCHY STATES AS CONFORMAL FIELD THEORY CORRELATORS

In this section, we review the procedure for expressing QH wave functions in a disk geometry as antisymmetrized sums of CFT correlators and emphasize several points that will be important for the generalization to the torus.

A. General considerations and the Laughlin state

Our aim is to relate the hierarchy wave functions on the torus to the holomorphic conformal blocks that build the correlation functions of vertex operators in certain CFT's. We first briefly recapitulate the construction in the plane outlined in Refs. 9 and 11.

The relevant CFT's have actions

$$S[\varphi_a] = \frac{g}{2\pi}\int d^2x \partial_\mu \varphi_a \partial^\mu \varphi_a, \tag{1}$$

where $\mu=1,2$, the metric is Euclidean, g is an overall normalization to be fixed later, and the boson fields $\varphi_a(z,\bar{z})$, with $z=x_1+ix_2=x+iy$, are compactified on circles with radii R_a. In loose analogy to the standard CF construction, we will need n fields to describe the CF state with n (partially) filled CF Landau levels.[27] Note that the compactification implies that φ_a are angular variables, so on manifolds of a higher genus, there are field configurations with nontrivial winding around the different handles.

The primary fields are given by the vertex operators

$$\hat{V}_{\mathbf{Q}}(z,\bar{z}) =: \exp\left[i\sum_{a=1}^{n} Q_a \varphi_a(z,\bar{z}) \right], \tag{2}$$

where the colons, which we suppress in the following, denote normal ordering, and $\mathbf{Q}$ is an n-dimensional charge vector:

$$\mathbf{Q} = \left(\frac{e_1}{R_1}, \ldots, \frac{e_n}{R_n} \right), \tag{3}$$

where e_i are integers.

On the plane, we choose the normalization g in the action so that the two-point function of the scalar fields is given by

$$\langle \varphi_a(z,\bar{z}) \varphi_b(w,\bar{w}) \rangle = -\delta_{ab} \ln|z-w|^2, \tag{4}$$

and hence a contraction of two vertex operators is given by

$$\langle \hat{V}_{\mathbf{Q}^{(i)}}(z,\bar{z}) \hat{V}_{\mathbf{Q}^{(j)}}(w,\bar{w}) \rangle = |z-w|^{2\mathbf{Q}^{(i)}\cdot\mathbf{Q}^{(j)}}, \tag{5}$$

for any charge vectors $\mathbf{Q}^{(i)}$ and $\mathbf{Q}^{(j)}$.

The statistics of the particles is coded in the operator product expansion (OPE) or, equivalently, in the two-point function of the corresponding holomorphic vertex operator $\hat{V}_{\mathbf{Q}}(z)$. Setting $i=j$ in Eq. (5), we find that the holomorphic part of the vertex operators with a $|\mathbf{Q}^{(i)}|^2$ odd has fermionic statistics. On the torus, the analysis is more involved, but the local properties of the vertex operators are the same as on the plane, as is evident from the pertinent OPE's. The statistics of the quasiholes[22] and quasielectrons[9] can be understood in a similar manner.

The QH wave functions can be obtained as correlators of the holomorphic part of the vertex operators. The simplest example is that of a Laughlin state at $\nu=1/q$. Here, we have only one field, φ, with radius $R^2=q$ and charge $e=q$, and we get

$$\left\langle \prod_{i=1}^{N} \hat{V}_{\sqrt{q}}(z_i) \mathcal{O}_{bg} \right\rangle = \Phi \prod_{i<j} (z_i - z_j)^q \exp\left(-\frac{1}{4\ell^2} \sum_i |z_i|^2 \right) \equiv \Phi \psi_{1/q}(z_i), \tag{6}$$

where $\mathcal{O}_{bg}$ is a constant neutralizing background and Φ is a singular phase factor that can be properly defined by replacing the continuous background charge by a lattice of thin flux tubes. Then, $\Phi=\Phi[\phi_i(\vec{n}_\alpha)]$, where $\phi_i(\vec{n}_\alpha)$ is the relative angle between the coordinate z_i and the lattice vector $\vec{n}_\alpha$. For the details of this procedure, which can be generalized to the full Jain series, see Appendix A in Ref. 9.

Alternatively, we can recover the wave functions by factoring the full correlation function into a holomorphic and an antiholomorphic part, also called conformal blocks, each accompanied by the square root of the nonfactorizable exponential factor,

$$\begin{aligned} \left\langle \prod_{i=1}^{N} \hat{V}_{\sqrt{q}}(z_i,\bar{z}_i) \mathcal{O}_{bg} \right\rangle &= \prod_{i<j} |z_{ij}|^{2q} \exp\left(-\frac{1}{2\ell^2} \sum_i |z_i|^2 \right) \\ &= \left(\prod_{i<j} z_{ij}^q \exp\left(-\frac{1}{4\ell^2} \sum_i |z_i|^2 \right) \right)^\star \\ &\quad \times \left(\prod_{i<j} z_{ij}^q \exp\left(-\frac{1}{4\ell^2} \sum_i |z_i|^2 \right) \right) \\ &\equiv [\psi_{1/q}(z_i)]^\star \psi_{1/q}(z_i), \end{aligned} \tag{7}$$

where $z_{ij}=z_i-z_j$. It is this latter procedure that will generalize to the torus. The background charge must be included to obtain a nonvanishing correlator. This will be discussed in detail in Sec. III. For now, it suffices to mention that it gives rise to the correct nonholomorphic dependence, and nothing more, as it does in the disk geometry.

On the torus, correlation functions will not factorize as in Eq. (7) but are given by a (generally infinite) sum over such terms. For a special class of CFT's, this infinite sum can be rewritten as a finite sum over extended conformal blocks. Such theories are called rational CFT's.[23] For a bosonic action of type (1), this requires that the radii R_a are of the form

$R_a^2=2p/p'$, with p and p' relatively prime. This particular rational CFT is called a rational torus. Clearly, the Laughlin state considered above falls into this class, as do the hierarchy states that are discussed in the following section.

In a general case, we will thus extract not only a single wave function, but a whole set. This provides an important consistency check on our method since the degeneracy of a lowest Landau level (LLL) state on the torus is known from the general symmetry analysis of Haldane.[24] The degeneracy of a state with filling fraction $\nu=p/q$ includes a factor of 4 related to choosing periodic or antiperiodic boundary conditions along the cycles of the torus and a q-fold degeneracy related to the position of the center of mass. As we will see, our construction precisely recovers this degeneracy.

B. Hierarchy wave functions

The hierarchy wave functions at level n, obtained by successive condensations of quasiparticles only, are described by a set of integers $\{k_1,\ldots,k_n\}$ related to the densities of the quasiparticle condensates. In the CFT scheme of Ref. 11, these wave functions are constructed from correlators involving n different vertex operators, V_α. They depend on the boson fields φ_a, $a=1,\ldots,\alpha$, and are constructed recursively by

$$V_1 = e^{i\gamma_1\varphi_1}$$

$$V_{\alpha+1} = \partial_z V_\alpha e^{-i\varphi_{\alpha'}/\gamma_\alpha} e^{i\gamma_{\alpha+1}\varphi_{\alpha+1}}, \quad \alpha=1,\ldots,n-1 \tag{8}$$

where $\gamma_1=\sqrt{2k_1+1}$, $\gamma_{\alpha+1}=\sqrt{2k_{\alpha+1}-\gamma_\alpha^{-2}}$, and $\varphi_{\alpha+1}$ is a new bosonic field. The electrons are divided into n sets I_α of size M_α, and the wave function is given by

$$\Psi = \mathcal{A}\left\langle \prod_{\alpha=1}^{n} \prod_{i_\alpha\in I_\alpha} V_\alpha(z_{i_\alpha}) \right\rangle, \tag{9}$$

where $\mathcal{A}$ denotes antisymmetrization and $\langle\cdots\rangle$ the correlation function in a suitable background field. M_α are determined recursively, as explained in Ref. 12. The filling factor of Ψ, $\nu_n=p_n/q_n$, is obtained recursively from

$$\nu_n = \frac{p_n}{q_n} = \frac{2k_n p_{n-1} - p_{n-2}}{2k_n q_{n-1} - q_{n-2}}, \tag{10}$$

with the initial conditions $q_0=p_0=1$ and the Laughlin series given by $\nu_1=1/(2k_1+1)$, with the k_1 integer. Taking $k_j=1$ amounts to having a maximal density in the jth condensate, and the Jain series $\nu_n=n/(2nk_1+1)$ corresponds to choosing $k_2=\cdots=k_n=1$.

To evaluate the correlators of operators (8) on the torus, we will use a different representation of the vertex operators, where the charge vectors are explicit,

$$V_{\mathbf{Q}^{(\alpha,n)}}(z,\bar{z}) = \mathcal{D}^{(\alpha-1)}\hat{V}_{\mathbf{Q}^{(\alpha,n)}}. \tag{11}$$

Here, $\mathcal{D}^{(\alpha)}$ are derivative operators to be discussed below. A vertex operator without a hat is a descendant, whereas a hat marks a primary field. The charge vectors $\mathbf{Q}^{(\alpha,n)}$ are not uniquely defined. We will mostly use a parametrization where the hierarchy construction is manifest,

$$\mathbf{Q}^{(\alpha,n)} = \left(\frac{c_1}{R_1},\ldots,\frac{c_{\alpha-1}}{R_{\alpha-1}},\frac{q_\alpha}{R_\alpha},0,\ldots,0\right). \tag{12}$$

The CFT charges c_j and q_j and the compactification radii R_j are connected to the denominators of the filling fractions by

$$c_j = q_j - q_{j-1}, \quad R_j^2 = q_j q_{j-1}. \tag{13}$$

For the Jain series, for example, all charges are equal to $2k_1$. When there is no ambiguity, we suppress the level index n on the charge vector and write just $\mathbf{Q}^{(\alpha)}$.

Other choices of basis sets represent the FQH state equally well as long as the bosonic fields, $\varphi_1,\ldots,\varphi_n$, have rational R_i^2 and the inner product of two vectors is unchanged: $\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\alpha')}=\delta_{\alpha\alpha'}+2k_1+\Sigma_{j=2}^{\alpha}2(k_j-1)$, for $\alpha\leqslant\alpha'$. In the special case of the Jain series, this relation simplifies to $\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\alpha')}=\delta_{\alpha\alpha'}+2k_1$.[9] This is discussed in more detail in Sec. V.

For later computations in Sec. IV C, we will also need the following relation between the number of states N_s and the charge vectors:

$$\forall\,\alpha: \sum_{\beta=1}^{n} \mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\beta)}M_\beta = N_s, \tag{14}$$

Together with $\Sigma_\beta M_\beta=N=\nu N_s$, this gives a consistency condition on N_s.

III. QUANTUM HALL STATES ON THE TORUS: GENERAL CONSIDERATIONS

In this section, we describe the techniques we use to construct the hierarchy wave functions on the torus.

A. Lowest Landau level on the torus

We consider a system of N charges $-e$ on the torus with periods L_1 and L_2. In the Landau gauge, a homogeneous external magnetic field perpendicular to the surface is described in terms of a vector potential, $\mathbf{A}=By\hat{\mathbf{x}}$. Most of the time, we will set the magnetic length equal to unity, $\ell=\sqrt{\hbar c/eB}=1$.

In the presence of an external magnetic field, consistent periodic boundary conditions can only be enforced up to a gauge transformation. A natural way to do this is to use the magnetic translation operators[24]

$$t(\mathbf{l}) = e^{\mathbf{l}\cdot(\nabla-i\mathbf{A})-i\mathbf{l}\times\mathbf{x}}, \tag{15}$$

where $\mathbf{l}$ parametrizes the translation, and $\mathbf{x}$ the electron coordinate. In the Landau gauge, we define

$$t_1 \equiv t\left(\frac{L_1}{N_s}\hat{\mathbf{x}}\right) = e^{(L_1/N_s)\partial_x}, \quad t_2 \equiv t\left(\frac{L_2}{N_s}\hat{\mathbf{y}}\right) = e^{(L_2/N_s)(\partial_y+ix)}, \tag{16}$$

where $N_s=L_1L_2/2\pi\in\mathbb{Z}$ is the number of flux quanta penetrating the surface of the torus. The translations $t_1^{N_s}$ and $t_2^{N_s}$ commute. Hence, periodic boundary conditions can be consistently formulated as

$$t_\mu^{N_s}\psi = \exp(i\phi_\mu)\psi, \quad \mu = 1,2, \tag{17}$$

where ϕ_μ can be interpreted as solenoid fluxes through the handles of the torus. Once the fluxes are fixed, the same boundary conditions apply for all states in the Hilbert space of the system.

A complete set of commuting operators for a $\nu=p/q$ QH system, with p and q relative primes, is given by $\{H,T_1,T_2^q\}$, where $T_\mu=\Pi_{i=1}^N t_{\mu,i}$ are center-of-mass translations and H is a translationally invariant electron-electron interaction. The states are labeled by $\{E,K_1,K_2\}$ where the quantum numbers K_α of the system relate to the eigenvalues of T_1 and T_2^q, which are given by $e^{2\pi i K_\alpha/N_s}$.

The lowest Landau level wave functions, in the Landau gauge, on the torus have many special properties discussed, for instance, in Refs. 24 and 25. They all factorize as

$$\psi(z_1,\ldots,z_N) = f(z_1,\ldots,z_N)\exp\left(-\sum_{k=1}^{N} y_k^2/2\right), \tag{18}$$

where f is holomorphic in all its arguments and is given in terms of generalized ϑ functions

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{k=-\infty}^{\infty} e^{i\pi\tau(k+a)^2}e^{2\pi i(k+a)(z+b)}. \tag{19}$$

For example, a Jastrow-type factor $\Pi_{i<j}(z_i-z_j)^q$ on the plane becomes $\Pi_{i<j}\vartheta_1\left(\frac{z_i-z_j}{L_1}\middle| i\frac{L_2}{L_1}\right)^q$ on the torus, where ϑ_1 corresponds to $a=b=1/2$.

From the boundary conditions in Eq. (17), one derives the corresponding conditions on the holomorphic functions f, as will be discussed in detail below.

Because of translational invariance, the center-of-mass (c.m.) dependence of the wave function can be separated,

$$f(z_1,\ldots,z_N) = F_{c.m.}(Z)f_{rel}(z_1,\ldots,z_N), \tag{20}$$

where $Z=\Sigma_i z_i/L_1$ and f_{rel} is independent of Z. While for the Laughlin states f_{rel} is uniquely determined (at least on the plane, sphere and torus) by its leading short-distance behavior, this is not the case for the Jain states, or more general hierarchy states. Of course, once the relative part is fixed, the degeneracy of the corresponding level is given by the number of center-of-mass functions compatible with the boundary conditions.

For hierarchy wave functions, we have not been able to explicitly separate the c.m. part even in the simple case of the Jain sequence. Here, the CFT techniques have proven to be extremely useful in that they allow for a direct construction of the wave functions without ever separating the c.m. part.

B. Background charge

A careful calculation shows that the correlators $\langle\Pi_i V_{\mathbf{Q}^{(i)}}(z_i,\bar{z}_i)\rangle$ vanish unless they satisfy the charge-neutrality condition $\Sigma_i\mathbf{Q}^{(i)}=0$. On the plane, there are two standard choices for the neutralizing background. For N particles with charge vectors $\mathbf{Q}_a$, one can either assume a compensating charge at infinity, $\mathcal{O}_{bg}=e^{-iN\Sigma_a Q_a\varphi_a(z_\infty,\bar{z}_\infty)}$ or a homogeneous droplet,

$$\mathcal{O}_{bg} = \prod_{a=1}^{n} \exp\left[-iR_a\rho_a\int d^2x\varphi_a(z,\bar{z})\right], \tag{21}$$

with $\rho_a=\rho_0/R_a^2$, $\rho_0=1/(2\pi\ell^2)$, being the density of the ath set. The latter method, which is more physical, is the one used in Eq. (7); it reproduces the correct nonholomorphic dependence of the LLL wave functions in the symmetric gauge. On the torus, the first choice is not possible at all; therefore, we will use the latter.

As in the plane, this homogeneous distribution cannot be realized using the operator content of the bosonic rational CFT, which implies that, in general, we cannot expect the conformal correlation functions to factorize into chiral components. On the plane, this did not cause any problems. On the contrary, the homogeneous background charge actually contributed the Gaussian nonholomorphic dependence needed in the LLL wave functions. Below, we find the same result on the torus.

C. Derivative operators $\mathcal{D}^{(k)}$

One of the basic difficulties in translating the hierarchy wave functions from the plane to the torus is related to the meaning of the derivative operators $\mathcal{D}^{(k)}$. In the plane, they are the holomorphic partial derivatives ∂_z^k, implying that the operators V_α are descendants of the primary fields in the theory. Using standard techniques, the corresponding correlation functions can be written as a product of derivatives acting on the correlation function of primary fields only. As discussed in Ref. 9, the derivatives act only on the polynomial part of the wave function, that is, on the conformal block, and not on the exponential factor. Thus, in the plane, the calculation of the wave functions reduces to the calculation of correlators of primary fields.

The origin of the derivative operators is easily understood in terms of composite fermions, where they are the remnants of the effective Landau level structure. Projection onto the lowest Landau level converts the $\bar{z}_i$'s, which occur in higher Landau levels, into derivatives ∂_i. In particular, this means that if a derivative is excluded, the wave function will vanish since it amounts to putting a particle in an already filled Landau level. For the general hierarchy state, the situation is more complicated, but numerical calculations indicate that the derivatives are also necessary here.

Turning to the torus, we have not been able to prove that the derivatives are needed in order to get nonvanishing wave functions. On the other hand, this is almost obvious since for large tori, the wave functions must be very similar to those on the plane, and these do need derivatives. We have also checked that the correlators of the primary fields do vanish after antisymmetrization in simple cases, e.g., for $\nu=2/5$.

A natural way to construct our wave functions on the torus would thus be to supplement an appropriately periodized version of the Jastrow factors occurring in the disk version, with derivatives and a suitably chosen c.m. part. Such an ansatz, however, does not work. A derivative acting only on the holomorphic part of a correlator destroys the quasiperiodicity of the wave function.[28]

We conclude that we must construct the $\mathcal{D}^{(k)}$'s in Eq. (11) from operators that preserve the Landau level index and the boundary conditions; i.e., they must commute with the one-particle Hamiltonian and the magnetic translations, $t_{\mu,i}^{N_s}$, around the cycles of the torus. Requiring, in addition, that they approach $a\partial+b\bar{\partial}+c$ (where a, b, and c are constants) on a large torus, leads to the finite translations in the x direction,

$$t_{1i}^{k} = e^{(kL_1/N_s)\partial_{x_i}} = e^{(2\pi k\ell^2/L_2)(\partial_i+\bar{\partial}_i)}, \tag{22}$$

where there are at most N_s distinct possibilities for the integer k. We note that these operators will preserve the quantum numbers of the many-body wave functions. A more thorough analysis shows that, in fact, only $\lceil N_s/2-1\rceil$—that is, $N_s/2-1$ if N_2 is an even number and $(N_s-1)/2$ if N_s is odd—are independent. Forming $\mathcal{D}^{(k)}$ from linear combinations of operators (22),

$$\mathcal{D}^{(k)} = \sum_{m=0}^{\lceil N_s/2-1\rceil} c_m^{(k)} t_1^m, \tag{23}$$

we can first compute the correlators of the primary fields and then act with the $\mathcal{D}^{(k)}$'s, just as in the plane. To get the correct limit for a large torus, it is natural to demand that $\mathcal{D}^{(k)} \to \partial^k$ as $N_s = L_1L_2/(2\pi\ell^2) \to \infty$, but this is not enough to determine the coefficients $c_m^{(k)}$ uniquely and, in fact, may not even be necessary.[29] For the states at levels 2 and 3 that we have tested, we used

$$\mathcal{D}^{(1)} = t_1 - 1,$$

$$\mathcal{D}^{(2)} = t_1^2 - 2t_1 + 1. \tag{24}$$

(The constant in $\mathcal{D}^{(k)}$ can be ignored as it gives no contribution to wave functions.) That the $c_m^{(k)}$'s cannot be determined uniquely might seem a serious drawback, but in Sec. VI where we present numerical tests of our wave funtions, we will see that for the $\nu=2/5$ state, the simplest choice, $\mathcal{D}^{(1)}=t_1$, already gives a good result. Furthermore, we show how this can be improved to yield a very good wave function by taking a linear combination of states where $\mathcal{D}^{(1)}=t_1^k$, with small k. We believe that adding such contributions from higher values of k is the torus counterpart of correcting the wave functions in the plane by adding contributions involving descendants of the primary fields defining the representative wave functions.[22]

D. Expressions for the correlators

Following the strategy outlined above, we now ignore the derivative operator in V_α and calculate the correlators of the corresponding primary operators, $\hat{V}_\alpha$, in the presence of a homogeneous background. Since the scalars φ_a are not coupled, the correlation function [Eq. (9)] factorizes and can be calculated using a straightforward generalization of known techniques (see e.g., Chap. 12 of Ref. 23),

$$\begin{aligned}
&\left\langle \prod_{\alpha=1}^{n} \prod_{i_\alpha\in I_\alpha} \hat{V}_{\mathbf{Q}^{(\alpha)}}(z_{i_\alpha},\bar{z}_{i_\alpha})\mathcal{O}_{bg} \right\rangle \\
&= \left\langle \prod_{\alpha=1}^{n} \prod_{i_\alpha\in I_\alpha} \exp\left[i\sum_a Q_a^{(\alpha)}\varphi_a(z_{i_\alpha},\bar{z}_{i_\alpha}) \right] \mathcal{O}_{bg} \right\rangle \\
&= \prod_{a=1}^{n} \left\langle \prod_{\alpha=1}^{n} \prod_{i_\alpha\in I_\alpha} e^{iQ_a^{(\alpha)}\varphi_a(z_{i_\alpha},\bar{z}_{i_\alpha})} \mathcal{O}_{bg} \right\rangle \\
&= |\psi_{jas}(z_{ij})|^2 \prod_{a=1}^{n} \mathcal{F}^{(a)}(Z^{(a)},\bar{Z}^{(a)}) f_{bg}^{(a)}(z_i,\bar{z}_i),
\end{aligned} \tag{25}$$

with

$$\psi_{jas}(z_{ij}) = \prod_{\gamma=1}^{n} \prod_{i_\gamma<j_\gamma\in I_\gamma} \vartheta_1(z_{i_\gamma j_\gamma}|\tau)^{\mathbf{Q}^{(\gamma)}\cdot\mathbf{Q}^{(\gamma)}} \prod_{\alpha<\beta}^{n} \prod_{\substack{i_\alpha\in I_\alpha \\ i_\beta\in I_\beta}} \vartheta_1(z_{i_\alpha i_\beta}|\tau)^{\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\beta)}}, \tag{26}$$

$$\mathcal{F}^{(a)}(Z^{(a)},\bar{Z}^{(a)}) = \sum_{e,m=-\infty}^{\infty} \exp\left(-2\pi i\rho_a R_a \int \frac{d^2x}{L_1}[\alpha_{e,m}z - \bar{\alpha}_{e,m}\bar{z}] \right) e^{i\pi[\tau\alpha_{e,m}^2 - \bar{\tau}\bar{\alpha}_{e,m}^2]} e^{2\pi i(\alpha_{e,m}[Z^{(a)}/R_a] - \bar{\alpha}_{e,m}[\bar{Z}^{(a)}/R_a])} \tag{27}$$

and

$$f_{bg}^{(a)}(z_i,\bar{z}_i) = \prod_{\alpha=1}^{n} \prod_{i_\alpha\in I_\alpha} \exp\left[-\rho_a R_a Q_a^{(\alpha)} \times \int d^2x \ln\left| \vartheta_1\left(\frac{z-z_{i_\alpha}}{L_1}|\tau \right) \right|^2 \right], \tag{28}$$

where $z_{i_\alpha j_\beta} = (z_{i_\alpha} - z_{j_\beta})/L_1$, etc., and

$$Z^{(a)} = R_a \sum_{\alpha=1}^{n} Q_a^{(\alpha)} Z_\alpha = q_a Z_a + c_a \sum_{\alpha=a+1}^{n} Z_\alpha, \tag{29}$$

with $Z_\alpha = \Sigma_{i_\alpha\in I_\alpha} z_{i_\alpha}/L_1$ being the (dimensionless) c.m. coordinate of the electrons in set I_α. We also introduced the notation $\tau = iL_2/L_1$ for the modular parameter describing the torus as well as $\alpha_{e,m} = e/R_a + mR_a/2$ and $\bar{\alpha}_{e,m} = e/R_a - mR_a/2$, where e and m are integers, to parametrize the electric and magnetic sectors of the CFT Hilbert space. Note that $Z^{(a)}$

depends on the filling fraction ν_n. In order to avoid confusion, we will often indicate the level of hierarchy by the subscript n, $Z_n^{(a)} \equiv Z^{(a)}$. When an explicit expression is needed, we will instead denote it by the filling fraction, e.g., $Z_{2/5}^{(a)}$.

The first exponential in Eq. (27) comes from the effect of the background charge for the different fields and vanishes if the integration domain is chosen to be $\int d^2x \equiv \int_{-L_1/2}^{L_1/2} dx \int_{-L_2/2}^{L_2/2} dy$. The integral in Eq. (28) can be calculated exactly, and by using the same integration domain, we obtain

$$I(z,\bar{z}) = \int d^2x' \ln \left| \vartheta_1\left(\frac{z'-z}{L_1}|\tau\right)\right|^2 = I(0,0) + 2\pi y^2. \tag{30}$$

In Appendix A, we show that a different choice of integration domain only amounts to a coordinate shift in the wave function. This, however, is not obvious but is a rather nontrivial consequence of the homogeneity of the states.

Substituting Eq. (30) into Eq. (28) yields, up to a constant factor,

$$f_{bg}^{(a)}(z_i,\bar{z}_i) = \prod_{\alpha=1}^{n} \prod_{i_\alpha \in I_\alpha} e^{-2\pi R_a \rho_a Q_a^{(\alpha)} y_{i_\alpha}^2}. \tag{31}$$

Finally, we can use the relation $\Sigma_{a=1}^n Q_a^{(\alpha)}/R_a = 1$, which simply expresses that all electrons have a unit charge[9] to get

$$\prod_a f_{bg}^{(a)} = \exp\left(-\sum_{k=1}^{N} y_k^2/\ell^2\right), \tag{32}$$

which is exactly the nonholomorphic Gaussian factor appropriate to the Landau gauge.

Since the background charge does not alter the form of the charge-lattice summation, the conformal blocks have the same structure as in a rational CFT. We stress that because of the Gaussian factor f_{bg}, and that alone, the terms in the sum that give the full correlator cannot be factorized into holomorphic and antiholomorphic blocks as it would should the operator content be purely that of a rational CFT. As in Eq. (7), we extract the QH wave function as the holomorphic conformal block times the square root of the Gaussian factor. In the following, we assume a homogeneous neutralizing background and only write the fully (anti)holomorphic blocks omitting the Gaussian factor.

E. Charge-lattice sums

In order to diagonalize magnetic translations in the Hilbert space spanned by the holomorphic conformal blocks, it is useful to simplify the charge-lattice sum. If the compactification radius is of the form $R_a^2 = 2p/p'$, then we can express the infinite sum in Eq. (27) as a finite sum over extended conformal blocks (see Appendix B)

$$\sum_{e,m} e^{i\pi\tau\alpha_{e,m}^2} e^{-i\pi\bar{\tau}\bar{\alpha}_{e,m}^2} e^{2\pi i[\alpha_{e,m}Z^{(a)} - \bar{\alpha}_{e,m}\bar{Z}^{(a)}]/R_a} = \sum_{r=0}^{p'-1} \sum_{s=0}^{2p-1} \mathcal{F}_{r,s}(Z^{(a)}) \bar{\mathcal{F}}_{-r,s}(\bar{Z}^{(a)}), \tag{33}$$

where the $2pp'$ functions

$$\mathcal{F}_{r,s}(Z^{(a)}) = \sum_{k=-\infty}^{\infty} e^{i\pi\tau(2pp'k + rp + sp')^2/2pp'} e^{(i\pi/p)(2pp'k+rp+sp')Z^{(a)}} \tag{34}$$

span the Hilbert space of the center-of-mass motion. Under single-particle lattice translations, they transform as follows:

$$\frac{\mathcal{F}_{r,s}(Z^{(a)}+c)}{\mathcal{F}_{r,s}(Z^{(a)})} = (-1)^{cr} e^{2\pi i c s/R_a^2},$$

$$\frac{\mathcal{F}_{r,s}(Z^{(a)}+c\tau)}{\mathcal{F}_{r,s+c}(Z^{(a)})} = e^{-i\pi\tau c^2/R_a^2} e^{-2\pi i c Z^{(a)}/R_a^2}. \tag{35}$$

Here, c are the integers related to the charges in the charge vectors [Eq. (29)].

IV. QUANTUM HALL STATES ON THE TORUS: EXPLICIT CONSTRUCTIONS

We now have all the pieces needed to extract the wave functions—except for one essential ingredient: If we extract the basis functions ψ_s from the correlators using the recipe $\Sigma_s[\psi_s(z_i)]^\star \psi_s(z_i)$ discussed in Sec. II, these will not satisfy the boundary conditions [Eq. (17)]. In this section, we describe in detail how to construct the correct linear combinations for the hierarchy states at filling fractions ν_n [Eq. (10)].

We start by constructing the wave functions explicitly in two simple cases: First, the Laughlin state at $\nu = 1/(2k+1)$ and then the simplest level 2 state, the $\nu = 2/5$, in the Jain sequence. The reader who carefully studies these examples should get a pretty good idea of the strategy for attacking the general case, the details of which are given in the last subsection.

A. Laughlin state

As already discussed, the Laughlin wave function for $\nu = 1/q$ on the torus[20] can be extracted from the N-point correlation function of the vertex operators $V_{\mathbf{Q}^{(1)}}$ depending on a single field φ_1 compactified on a circle with radius $R_1^2 = q$ (i.e., $p=q$ and $p'=2$) and with charge vector $\mathbf{Q}^{(1)} = q/R_1 = \sqrt{q}$. In this case, there is only one condensate and $Z^{(1)} = 3Z$.

Using Eqs. (25) and (33), we can express the holomorphic wave functions in a basis given by the chiral conformal blocks,

$$\psi_{r,s}(z_i) = \prod_{i<j} \vartheta_1(z_{ij}|\tau)^q \exp\left(-\frac{1}{2}\sum_k y_k^2\right) \mathcal{F}_{r,s}(Z^{(1)}). \tag{36}$$

A maximal linearly independent set of states is obtained with $r = 0,1$ and $s = 0,1,\ldots,2q-1$. The number of conformal

blocks is $2pp'=4q$, which is the expected number of degenerate states—with a factor q coming from the c.m. degeneracy and a factor of 2×2 from the different boundary conditions [Eq. (17)]. The q-fold degenerate multiplet of Laughlin wave functions is given by the linear combinations of the basis states [Eq. (36)], which transform irreducibly under magnetic lattice translations and satisfy the same solenoid flux conditions.

To find the physical states, we have to diagonalize the action of the single-particle magnetic translation operators, $t_{\alpha,i}^{N_s}$, on the full wave functions. To make the analysis more transparent, however, let us first summarize the relevant transformation properties of the holomorphic components of the basis states. The odd Jacobi theta function transforms as

$$\vartheta_1(z+1|\tau) = -\vartheta_1(z|\tau),$$

$$\vartheta_1(z+\tau|\tau) = -e^{-i\pi\tau}e^{-2\pi iz}\vartheta_1(z|\tau). \tag{37}$$

The transformation properties of the holomorphic center-of-mass functions can be read off from Eq. (35) and are given by

$$\mathcal{F}_{r,s}(Z^{(1)}+q) = (-1)^r\mathcal{F}_{r,s}(Z^{(1)}),$$

$$\mathcal{F}_{r,s}(Z^{(1)}+q\tau) = e^{-i\pi\tau q}e^{-2\pi iZ^{(1)}}\mathcal{F}_{r,s+q}(Z^{(1)}). \tag{38}$$

The functions should also satisfy the periodicity conditions

$$\mathcal{F}_{r,s+2q}(Z^{(1)}) = \mathcal{F}_{r,s}(Z^{(1)}),$$

$$\mathcal{F}_{r+4,s}(Z^{(1)}) = \mathcal{F}_{r,s}(Z^{(1)}). \tag{39}$$

From these relations, we can work out the action of the magnetic translations on the basis functions [Eq. (36)]; in the x direction, this gives

$$\frac{t_{1,i}^{N_s}\psi_{r,s}(z_i)}{\psi_{r,s}(z_i)} = \prod_{i<j}\frac{\vartheta_1(z_{ij}+1|\tau)^q}{\vartheta_1(z_{ij}|\tau)^q}\frac{\mathcal{F}_{r,s}(Z^{(1)}+q)}{\mathcal{F}_{r,s}(Z^{(1)})} = (-1)^{q(N-1)+r}, \tag{40}$$

and in the y direction

$$\frac{t_{2,i}^{N_s}\psi_{r,s}(z_i)}{\psi_{r,s+q}(z_i)} = \prod_{j\neq i}\frac{\vartheta_1(z_{ij}+\tau|\tau)^q}{\vartheta_1(z_{ij}|\tau)^q}\frac{\mathcal{F}_{r,s}(Z^{(1)}+q\tau)}{\mathcal{F}_{r,s+q}(Z^{(1)})}e^{i\,\mathrm{Im}\,\tau z_i+(1/2)(\mathrm{Im}\,\tau)^2} = (-1)^{q(N-1)}, \tag{41}$$

where, in the last step, we used the identities $Nq=N_s$ and $L_1L_2=2\pi N_s$. This shows that $\psi_{r,s}$ are $t_{1,i}^{N_s}$ eigenstates with eigenvalues independent of s, while $t_{2,i}^{N_s}$ maps the function $\psi_{r,s}(z_i)$ into the linearly independent function $\psi_{r,s+q}(z_i)$. Since, however, both transformations are independent of s, we can, in a unique way, satisfy the boundary conditions [Eq. (17)] by forming the linear combinations,

$$\mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)}) = \mathcal{F}_{r,\bar{s}}(Z^{(1)}) + (-1)^t\mathcal{F}_{r,\bar{s}+q}(Z^{(1)}) = \sum_k (-1)^{kt}e^{i\pi\tau q(k+a_1)^2}e^{2\pi i(k+a_1)qZ}, \tag{42}$$

where $a_1=\frac{\bar{s}}{q}+\frac{r}{2}$, $t=0,1$ and $\bar{s}=0,\ldots,q-1$, which amounts to a change of basis for the conformal blocks spanning the Hilbert space of the c.m. wave function. Thus, defining

$$\psi_{r,t,\bar{s}}(z_i) = \prod_{i<j}\vartheta_1(z_{ij}|\tau)^q\mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)})\exp\left(-\frac{1}{2}\sum_k y_k^2\right), \tag{43}$$

we finally have

$$t_{1,i}^{N_s}\psi_{r,t,\bar{s}}(z_i) = (-1)^{q(N-1)+r}\psi_{r,t,\bar{s}}(z_i),$$

$$t_{2,i}^{N_s}\psi_{r,t,\bar{s}}(z_i) = (-1)^{q(N-1)+t}\psi_{r,t,\bar{s}}(z_i). \tag{44}$$

Thus, the states $\psi_{r,t,\bar{s}}$ are eigenstates of $t_{\alpha,i}^{N_s}$, $\alpha=1,2$. For a fixed number of particles, the four different choices of the solenoid fluxes in Eq. (17) precisely correspond to the four combinations of the quantum numbers r and t, while the quantum number $\bar{s}$ determines the position of the c.m. For a translationally invariant Hamiltonian, this implies a q-fold degeneracy.

The many-body quantum numbers are related to the c.m. translations $T_1=\Pi_{i=1}^N t_{1,i}$ and $T_2^q=\Pi_{i=1}^N t_{2,i}^q$, which, together with the Hamiltonian, form a maximal set of commuting operators. The operator T_1 acts only on the holomorphic c.m. piece, and we get

$$T_1\mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)}) = \mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)}+L_1) = (-1)^r e^{2\pi is/q}\mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)}). \tag{45}$$

Compared with the definition of the quantum numbers K_i, given by $T_i\psi=e^{2\pi iK_i/N_s}\psi$, it follows that $K_1=(rN_s/2+N\bar{s})\mathrm{mod}\,N_s$. T_2 acts on both the Gaussian and the holomorphic c.m. piece. We get

$$T_2^q\left[\exp\left(-\frac{1}{2}\sum_k y_k^2\right)\mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)})\right] = (-1)^t\exp\left(-\frac{1}{2}\sum_k y_k^2\right)\mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)}), \tag{46}$$

and hence $K_2=(tN_s/2)\mathrm{mod}\,N_s$. This analysis establishes that states with unequal $\bar{s}$ have different quantum numbers and are therefore orthogonal. We conclude, in agreement with Ref. 20, that the degeneracy of Laughlin's states, as obtained from conformal correlators, is given by the denominator of the filling factor.

Let us summarize the result for the case $\phi_a=0$, where $r=t=(N_s-q)\mathrm{mod}\,2$. In order to write the result in a more conventional notation, we note that the holomorphic c.m. function can be written in terms of the Jacobi theta function [Eq. (19)]. A comparison with Eq. (42) gives

$$\mathcal{H}_{r,t,\bar{s}}^{(1)}(Z^{(1)}) = \vartheta\begin{bmatrix}(N_s-q)/2+\bar{s}/q\\(N_s-q)/2\end{bmatrix}(qZ|q\tau). \tag{47}$$

In this notation, the q-fold degenerate multiplet of Laughlin wave functions is given by

$$\psi_{\bar{s}}(z_i) = \prod_{i<j}^{N} \vartheta_1(z_{ij}|\tau)^q \exp\left(-\frac{1}{2}\sum_k y_k^2\right) \times \vartheta\begin{bmatrix} (N_s-q)/2+\bar{s}/q \\ (N_s-q)/2 \end{bmatrix}(qZ|q\tau), \tag{48}$$

with $\bar{s}=0,\ldots,q-1$. Up to an overall constant, the result is identical to that obtained in Ref. 20.

B. $\nu=2/5$ state

As a first example of a second level hierarchy state, we now explicitly compute the wave function for the filling fraction $\nu=2/5$. We use the same approach as for the Laughlin wave functions: The holomorphic part of the correlator, together with the square root of the Gaussian, yields a set of basis states. We construct all linear combinations that are eigenfunctions of the single-particle magnetic translation operators $t_{1,i}^{N_s}$ and $t_{2,i}^{N_s}$. For simplicity, we set both solenoid fluxes to zero. This yields five candidate wave functions with T_1 eigenvalues $e^{2\pi i n\nu}$.

The correlator is built from vertex operators constructed from two bosonic fields with radii $R_1^2=3$ and $R_2^2=15$. Since this is a Jain state, corresponding to two completely filled CF Landau levels, the sets contain equal number of electrons so that $M_1=M_2=N/2$. In the hierarchy picture, this amounts to having a maximally dense condensate of quasielectrons on top of the $\nu=1/3$ parent state.

The charge vectors are given by

$$\mathbf{Q}^{(1)} = \left(\frac{3}{\sqrt{3}},0\right),$$

$$\mathbf{Q}^{(2)} = \left(\frac{2}{\sqrt{3}},\frac{5}{\sqrt{15}}\right), \tag{49}$$

and we get $Z^{(1)}=3Z_1+2Z_2$ and $Z^{(2)}=5Z_2$, where Z_1 and Z_2 are the c.m. coordinates of the two sets, respectively. Following the strategy outlined in Sec. IV A, we ignore the derivative operator in V_2 and calculate the correlators of the corresponding primary operators. The pertinent correlation function decomposes as

$$\left\langle \prod_{j\in I_1} \hat{V}_1(z_j,\bar{z}_j) \prod_{a\in I_2} \hat{V}_2(z_a,\bar{z}_a) \right\rangle = \left\langle \prod_{j\in I_1} e^{i(3/\sqrt{3})\varphi_1(z_j,\bar{z}_j)} \prod_{a\in I_2} e^{i(2/\sqrt{3})\varphi_1(z_a,\bar{z}_a)} \right\rangle \times \left\langle \prod_{a\in I_2} e^{i(5/\sqrt{15})\varphi_2(z_a,\bar{z}_a)} \right\rangle. \tag{50}$$

The correlator involving φ_1 contributes the following holomorphic factor:

$$\prod_{i<j\in I_1} \vartheta_1(z_{ij}|\tau)^3 \prod_{a<b\in I_2} \vartheta_1(z_{ab}|\tau)^{4/3} \times \prod_{i\in I_1, a\in I_2} \vartheta_1(z_{ia}|\tau)^2 \mathcal{G}^1_{r,t,s}(Z^{(1)}), \tag{51}$$

where $\mathcal{G}^1_{r,t,s}(z)\equiv\mathcal{H}^{(1)}_{r,t,s}(z)$. This notation is chosen to be consistent with the general hierarchy, where $\mathcal{H}^{(n)}_{r,t,s}$ is used for the correct c.m. dependence at level n. Whenever convenient, we will suppress the r and t dependence and drop the bar on $\bar{s}$, still keeping track of the proper range of this index.

The second correlator contributes a factor, which can be parametrized as

$$\prod_{a<b\in I_2} \vartheta_1(z_{ab}|\tau)^{5/3} \mathcal{G}^2_s(5Z_2), \tag{52}$$

with

$$\mathcal{G}^2_s(5Z_2) = \sum_k (-1)^{tk} e^{i\pi\tau 15(k+a_2)^2} e^{2\pi i(k+a_2)5Z_2} \tag{53}$$

and $a_2=\frac{s_2}{15}+\frac{r}{2}$. The basis set of wave functions is then obtained as a product of the conformal blocks [Eqs. (51) and (52)]. The derivatives commute with the magnetic translation operators, and Eq. (17) must be fulfilled for all divisions into sets separately. Therefore, we can consider the functions

$$\psi = \psi_{jas}\mathcal{H}^{(2)}(Z^{(1)},Z^{(2)})\exp\left(-\frac{1}{2}\sum_k y_k^2\right), \tag{54}$$

instead of the full wave function without loss of generality. The c.m. part, $\mathcal{H}^{(2)}(Z^{(1)},Z^{(2)})$, is a sum of products $\mathcal{G}^1_{s_1}(Z^{(1)})\mathcal{G}^2_{s_2}(Z^{(2)})$ and ψ_{jas} denotes the Jastrow factor expected from the result on the plane. Even though $\mathcal{H}^{(2)}\times(Z^{(1)},Z^{(2)})$ depends on the c.m. of the sets and, therefore, on both total c.m. and relative coordinates, we will still call it the c.m. function in the following sections.

Note that we choose to express the correlators in the basis defined by Eq. (42) rather than the one obtained directly from Eq. (36). This will be very convenient since by construction the basis we use already incorporates the correct boundary conditions for the first set. As we will see, a similar thing happens for a general hierarchy state, where a proper choice of conformal blocks will automatically impose the correct transformation properties for all but the last set of particles.

As for the Laughlin states, we proceed by demanding $t_{\alpha,i}^{N_s}$ to be diagonal (with eigenvalues +1) on the wave function. As the transformation properties of both the Jastrow factors and the Gaussian are known, we focus only on the c.m. dependence and derive its properties under single-particle translations of $z_i\in I_\alpha$,

$$\frac{\mathcal{H}^{(2)}(Z_\alpha+1)}{\mathcal{H}^{(2)}(Z_\alpha)} = e^{i\pi(N_s-\kappa_\alpha)},$$

$$\frac{\mathcal{H}^{(2)}(Z_\alpha+\tau)}{\mathcal{H}^{(2)}(Z_\alpha)} = e^{i\pi(N_s-\kappa_\alpha)}e^{-i\pi\tau 3}\exp\left(-2\pi i\mathbf{Q}^{(\alpha)}\sum_\beta \mathbf{Q}^{(\beta)}Z_\beta\right). \tag{55}$$

We suppress the dependence on all c.m. coordinates but the translated one in this and the following section. We also introduce $|\mathbf{Q}^{(\alpha)}|^2=\kappa_\alpha$ as an abbreviation. Note, that κ_α is odd for all α's since the vertex operators are fermionic. Thus, all electrons obey the same boundary conditions, independent of which set they are in.

Since $\mathcal{H}^{(2)}$ is a linear combination of products $\mathcal{G}^1_{s_1}\mathcal{G}^2_{s_2}$, we first determine their properties under single-particle translations. As already pointed out, the former has, by construction, the correct properties for translations of particles in the first set, but not for those in the second,

$$\frac{\mathcal{G}^1_{s_1}(Z^{(1)}+c_1)}{\mathcal{G}^1_{s_1}(Z^{(1)})}=e^{2\pi i a_1 2},$$

$$\frac{\mathcal{G}^1_{s_1}(Z^{(1)}+c_1\tau)}{\mathcal{G}^1_{s_1+2}(Z^{(1)})}=e^{-2\pi i(2Z_1+(4/3)Z_2)}e^{-i\pi\tau(4/3)}. \tag{56}$$

Comparing these expressions with those of $\mathcal{G}^2_{s_2}(Z^{(2)})$,

$$\frac{\mathcal{G}^2_{s_2}(Z^{(2)}+q_2)}{\mathcal{G}^2_{s_2}(Z^{(2)})}=e^{2\pi i a_2 5},$$

$$\frac{\mathcal{G}^2_{s_2}(Z^{(2)}+q_2\tau)}{\mathcal{G}^2_{s_2+5}(Z^{(2)})}=e^{-2\pi i(5/3)Z_2}e^{-i\pi\tau(25/15)}, \tag{57}$$

we find that the coordinate dependent factors (independent of the free parameters s_1 and s_2) combine to give $2Z_1+3Z_2=\mathbf{Q}^{(2)}\cdot\Sigma_\alpha\mathbf{Q}^{(\alpha)}Z_\alpha$ and, thus, the correct factor to cancel the one from the relative part and the Gaussian. This is not a coincidence, as it may seem here, but a general property due to the construction of the vertex operators. Correct transformation properties along L_1 require special combinations of s_1 and s_2, so that $(2s_1+s_2)/3$ is an integer. Hence, we find that only the combinations where $-s_1+s_2=0 \bmod 3$ are consistent with the eigenvalues in Eq. (55). This reduces the number of "good" basis states from 45 to 15 ($\times 4$ for the different flux sectors). By taking $r=(N_s-\kappa_1)\bmod 2$, we obtain the correct sign in Eq. (55).

We can now use this reduced set to find the linear combinations that also transform correctly under $t^{N_s}_{2,i}$. It is easy to see that if a pair (s_1,s_2) satisfies $-s_1+s_2=3k$, then (s_1+2,s_2+5) does, too. Therefore, $t^{N_s}_{2,i}$ only maps functions in this reduced set into each other. Hence, we use the parametrization $(s_1,s_2)=(2l,5l+3s')$ and form the linear combination,

$$\mathcal{H}^{(2)}_{\bar{s}}(Z^{(1)},Z^{(2)})=\sum_{l=0}^{2}(-1)^{tl}\mathcal{G}^1_{2j}(Z^{(1)},Z^{(2)})\mathcal{G}^2_{5l+3s'}(Z^{(2)}), \tag{58}$$

which transforms correctly if t is chosen to be $t=N_s-\kappa_1$. Note that we use $\bar{s}=3s' \bmod 5$, $s'=0,\ldots,4$ to label the c.m. coordinate. That this is a natural choice is seen by the action of T_1 in Eq. (58),

$$\begin{aligned}T_1\mathcal{H}^{(2)}_{\bar{s}}(Z^{(1)},Z^{(2)})&=\sum_{l=0}^{2}(-1)^{tl}\mathcal{G}^1_{2l}(Z^{(1)})\mathcal{G}^2_{5l+3s'}(Z^{(2)})\\&\quad\times e^{2\pi i a_1(3+2)(N/2N_s)}e^{2\pi i a_2 5(N/2N_s)}\\&=\mathcal{H}^{(2)}_{\bar{s}}(Z^{(1)},Z^{(2)})e^{2\pi i(2/5)\bar{s}},\end{aligned} \tag{59}$$

where $\bar{s}$ naturally occurs multiplied with the filling fraction $\nu=2/5$, which amounts to $K_1=(2N\bar{s})\bmod N_2$. In addition, T_2 acting on $\mathcal{H}^{(2)}_{\bar{s}}$ only shifts $\bar{s}\to\bar{s}+1$. Thus, $\mathcal{H}^{(2)}_{\bar{s}}$ is invariant under T_2^5, as expected, and $K_2=0$. There is no overall sign, as both $\mathcal{G}^1$ and $\mathcal{G}^2$ pick up a factor $(-1)^t$ under $\bar{s}\to\bar{s}+5$.

In order to get the wave function, we need to reintroduce the derivatives and antisymmetrize over all possible divisions into the two sets:

$$\begin{aligned}\Psi_{2/5}=&\sum_{\substack{i_1<i_2<\cdots i_{N/2}\\a_1<a_2<\cdots a_{N/2}}}(-1)^{(\Sigma_j a_j)}\prod_k\mathcal{D}^{(1)}_{a_k}\prod_{i_j<i_l}\vartheta_1(z_{i_j i_l}|\tau)^3\\&\times\prod_{a_j<a_l}\vartheta_1(z_{a_j a_l}|\tau)^3\prod_{i_j,a_l}\vartheta_1(z_{i_j a_l}|\tau)^2\mathcal{H}^{(2)}_{\bar{s}}(Z^{(1)},Z^{(2)})\\&\times\exp\left(-\frac{1}{2\ell^2}\sum_k y_k^2\right),\end{aligned} \tag{60}$$

with $Z^{(1)}=3\Sigma_j z_{i_j}/L_1+2\Sigma_j z_{a_j}/L_1$ and $Z^{(2)}=5\Sigma_j z_{a_j}/L_1$. There are $\lceil N_s/2-1\rceil$ possible choices for the derivative operator for a fixed set of quantum numbers of $\Psi_{2/5}$. More comments on this can be found in Sec. VI where we test the candidate wave function numerically against exact diagonalization results.

C. General hierarchy state

In this section, we generalize the previous construction of the Laughlin and $\nu=2/5$ wave functions on the torus to the general class of filling fractions discussed in Sec. II B.

Recall that to construct the hierarchy states at level n, the electrons are divided into n sets I_α. The electrons in set I_α are represented by $V_{\mathbf{Q}^{(\alpha)}}$, giving a center-of-mass coordinate $Z_\alpha=\Sigma_{k\in I_\alpha}z_k/L_1$. Thus, the correlator we want to compute is of the form

$$\left\langle\prod_{\alpha=1}^{n}\prod_{k\in I_\alpha}\hat{V}_{\mathbf{Q}^{(\alpha)}}(z_k,\bar{z}_k)\right\rangle. \tag{61}$$

It factorizes into a product of correlators containing only one field, each of which can be calculated as in Sec. III. In the following, we use the same conventions as in the previous section. We denote the functions $\mathcal{G}^a_{r,t,s_a}$ as c.m. functions even though they depend on the c.m. of the sets and not the total c.m. For simplicity, the fluxes are set to zero, $\phi_1=\phi_2=0$, and we suppress the indices r and t as they are fixed by the boundary conditions.

The wave functions are extracted from Eq. (61) by exactly the same approach used in the Laughlin case and $\nu=2/5$. As in the latter, the wave function is an antisymmetric sum over the various ways to divide the electrons into the sets. Each summand must be an eigenfunction of the magnetic translation operators. This constraint is used to construct the suitable linear combinations of the conformal blocks for any given division. The basis states are given by

$$\psi_{r,t,\{s_a\}} = \prod_{\alpha=1}^{n} \prod_{i_\alpha<j_\alpha\in I_\alpha} \vartheta_1(z_{i_\alpha j_\alpha}|\tau)^{\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\alpha)}} \times \prod_{\alpha<\beta} \prod_{\substack{i_\alpha\in I_\alpha \\ j_\beta\in I_\beta}} \vartheta_1(z_{i_\alpha j_\beta}|\tau)^{\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\beta)}} \prod_{a=1}^{n} \mathcal{G}^a_{s_a}(Z_n^{(a)}) \times \exp\left(-\frac{1}{2}\sum_k y_k^2\right), \tag{62}$$

where we used the same conventions for $\mathcal{G}^j_{s_j}$, as in the 2/5 case,

$$\mathcal{G}^j_{s_j}(Z_n^{(j)}) = \sum_k (-1)^{tk} e^{i\pi\tau R_j^2(k+a_j)^2} e^{2\pi i(k+a_j)Z_n^{(j)}}, \tag{63}$$

with $a_j=\frac{s_j}{R_j^2}+\frac{r}{2}$ and $s_j=1,\ldots,4R_a^2-1$. Their transformation under translations along L_1 and L_2 follow directly from Eq. (35),

$$\frac{\mathcal{G}^j_{s_j}(Z_n^{(j)}+c)}{\mathcal{G}^j_{s_j}(Z_n^{(j)})} = e^{2\pi i a_j c},$$

$$\frac{\mathcal{G}^j_{s_j}(Z_n^{(j)}+c\tau)}{\mathcal{G}^j_{s_j+c}(Z_n^{(j)})} = e^{-2\pi i(c/R_j^2)Z_n^{(j)}} e^{-i\pi\tau(c^2/R_j^2)}, \tag{64}$$

and they have the periodicity property

$$\mathcal{G}^j_{s_j+R_j^2}(Z_n^{(j)}) = (-1)^t e^{-i\pi\tau R_j^2} e^{-2\pi i Z_n^{(j)}} \mathcal{G}^j_{s_j}(Z_n^{(j)}). \tag{65}$$

The functions $\mathcal{G}^j$ depend only on $Z_n^{(j)}=c_j\Sigma_{\alpha=j+1}^n Z_\alpha+q_jZ_j$ and are thus independent of the first $j-1$ c.m. coordinates. In addition, they are identical for the parent and daughter states, except that $Z_n^{(j)}$ must be replaced by $Z_{n+1}^{(j)}$. This yields a recursive construction of the full c.m. dependence $\mathcal{H}^{(n)}_{s_n}$. To be more specific, we will show by recursion that at filling fraction $\nu_n=p_n/q_n$, there are exactly q_n possible c.m. functions $\mathcal{H}^{(n)}_{\bar{s}_n}$ given by

$$\mathcal{H}^{(n)}_{\bar{s}_n}(Z_n^{(1)},\ldots,Z_n^{(n)}) = \sum_{l_2=0}^{q_1-1} \cdots \sum_{l_n=0}^{q_{n-1}-1} \prod_{a=1}^{n} (-1)^{t(l_2+\cdots+l_n)} \mathcal{G}^a_{s_a}(Z_n^{(a)}), \tag{66}$$

where

$$s_a = q_a l_a + \sum_{k=a+1}^{n} c_a l_k, \quad a<n,$$

$$s_n = q_n l_n + q_{n-1} s. \tag{67}$$

In analogy with the $\nu=2/5$ case, and for reasons that will become clear below, we choose $\bar{s}_n$ as the free parameter, with $\bar{s}_n=q_{n-1}s \bmod q_n$, $s=0,\ldots,q_n-1$.

We have already shown that the c.m. part of the Laughlin wave functions is of the form of Eq. (66). It remains to show that, assuming Eq. (66) for a parent state at level n, it follows that also the c.m. part of the daughter state at level $n+1$ obeys this relation. To go from level n to level $n+1$ in the hierarchy, we consider a quasiparticle density of $1/(2k_{j+1}+1)$ atop the parent state. The daughter state is then at filling fraction ν_{n+1} determined by Eq. (10). To construct the wave function, there is then an additional vertex operator V_{n+1}, with charge vector $\mathbf{Q}^{(n+1)}=\left(\frac{c_1}{R_1},\ldots,\frac{c_n}{R_n},\frac{q_{n+1}}{R_{n+1}}\right)$. Note that all charge vectors are now $(n+1)$-dimensional objects. The wave function will be of the form

$$\psi = \psi_{jas}(z_{ij}) \mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z_{n+1}^{(1)},\ldots,Z_{n+1}^{(n+1)}) \exp\left(-\frac{1}{2}\sum_i y_i^2\right), \tag{68}$$

where $\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}$ is some suitable linear combination of the products of $\mathcal{G}^a_{s_a}$.

Invariance under one-particle magnetic translations puts constraints on the wave function. As the Jastrow-type factor is known, we can compute which relations $\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}$ must satisfy under one-particle (normal) translations. Under the translations $Z_\alpha\to Z_\alpha+1$ and $Z_\alpha\to Z_\alpha+\tau$, $\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}$ transforms as

$$\frac{\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z_\alpha+1)}{\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z_\alpha)} = (-1)^{(N_s-\kappa_\alpha)},$$

$$\frac{\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z_\alpha+\tau)}{\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z_\alpha)} = (-1)^{(N_s-\kappa_\alpha)} \times \exp\left(-2\pi i \mathbf{Q}^{(\alpha)}\cdot\sum_\beta \mathbf{Q}^{(\beta)} Z_\beta\right) e^{-i\pi\tau\kappa_\alpha}, \tag{69}$$

where we use the same simplified notation as in Eq. (55): $\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z_\alpha+c)=\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z_{n+1}^{(1)},\ldots,Z_{n+1}^{(n+1)})$, but with Z_α replaced by $Z_\alpha+c$ in the $Z^{(i)}$'s. In deriving the second of these relations, we used Eq. (14). Equation (69) is valid for the c.m. dependent functions at all levels, in particular, also at level n.

The function $\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}$ is a linear combination of products: $\Pi_{j=1}^{n+1}\mathcal{G}^{(j)}_{s_j}(Z_{n+1}^{(j)})$. The task of finding the correct linear combination is simplified by using the result for the parent state. By construction, $\mathcal{H}^{(n)}_{\bar{s}_n}(Z_{n+1}^{(1)},\ldots,Z_{n+1}^{(n)})$ transforms correctly under translations of all c.m. coordinates Z_α but Z_{n+1}. For this last coordinate, note that for $a<n$, $\mathcal{G}^{(a)}$ depends in the same way on Z_{n+1} as on Z_n, as $Z_{n+1}^{(a)}=q_aZ_a+c_a(Z_{a+1}+\cdots+Z_n+Z_{n+1})$. This is not true for $\mathcal{G}^{(n)}_{s_n}$, but we can separate the difference by writing

$$\mathcal{G}^{(n)}_{s_n}(Z_{n+1}^{(n)}) = \mathcal{G}^{(n)}_{s_n}[q_n(Z_n+Z_{n+1})] f(Z_n,Z_{n+1}), \tag{70}$$

with

$$f(Z_n,Z_{n+1}) = \frac{\mathcal{G}^n_{s_n}(q_nZ_n + c_nZ_{n+1})}{\mathcal{G}^n_{s_n}(q_nZ_n + q_nZ_{n+1})}. \tag{71}$$

This implies that for a given set $(l_2,\ldots,l_n)$ in Eq. (66), the product $\mathcal{G}^1\times\cdots\times\mathcal{G}^n$ transforms in the same way under translations of Z_n and Z_{n+1}, except for the ratio [Eq. (71)]. The transformations of this ratio is, however, readily obtained from Eq. (63), and since it is independent of l_n, we can infer the following relation for translations along L_1:

$$\mathcal{H}^{(n)}_{\bar{s}_n}(Z_{n+1}+1) = e^{-2\pi i(\bar{s}/q_n)}\mathcal{H}^{(n)}_{\bar{s}_n}(Z_{n+1}), \tag{72}$$

and a slightly more complicated relation for translations along L_2,

$$\begin{aligned}\mathcal{H}^{(n)}_{\bar{s}_n}(Z_{n+1}+\tau) &= \exp\left[-2\pi i\sum_{\beta=1}^{n}\left(\mathbf{Q}^{(n+1)}\cdot\mathbf{Q}^{(\beta)}Z_\beta + \frac{c_\beta^2}{R_\beta^2}Z_{n+1}\right)\right]\\ &\times e^{-i\pi\tau[\kappa_n+(c_n^2-q_n^2)/R_n^2]}\\ &\times\sum_{l_2=0}^{q_1}\cdots\sum_{l_n=0}^{q_{n-1}-1}\prod_{a=1}^{n}(-1)^{t(l_2+\cdots+l_n)}\mathcal{G}^a_{s_a+c_a}(Z^{(a)}_{n+1})\\ &= \pm\exp\left[-2\pi i\sum_{\beta=1}^{n}\left(\mathbf{Q}^{(n+1)}\cdot\mathbf{Q}^{(\beta)}Z_\beta\right.\right.\\ &\left.\left.+\frac{c_\beta^2}{R_\beta^2}Z_{n+1}\right)\right]e^{-i\pi\tau[\kappa_n+(c_n^2-q_n^2)/R_n^2]}\mathcal{H}^{(n)}_{\bar{s}_n+c_n}(Z_{n+1}).\end{aligned} \tag{73}$$

The sign is calculable but is not needed in the following discussion. As q_n and q_{n+1} are relatively prime, $jc_n \bmod q_n$ runs through all possible values of q_n. Therefore, invariance under translations of the set I_{n+1} will require linear combinations of all q_n c.m. functions of level n.

To obtain an eigenfunction of $t_{\alpha i}^{N_s}$, $\mathcal{H}^{(n)}_{\bar{s}_n}$ must be combined with $\mathcal{G}^{n+1}_{s_{n+1}}(Z^{(n+1)}_{n+1})$, which transforms according to Eq. (64) with $c=q_{n+1}$,

$$\frac{\mathcal{G}^{n+1}_{s_{n+1}}(Z^{(n+1)}_{n+1}+q_{n+1})}{\mathcal{G}^{n+1}_{s_{n+1}}(Z^{(n+1)}_{n+1})} = e^{2\pi i(s_{n+1}/q_n)},$$

$$\frac{\mathcal{G}^{n+1}_{s_{n+1}}(Z^{(n+1)}_{n+1}+q_{n+1}\tau)}{\mathcal{G}^{n+1}_{s_{n+1}+q_{n+1}}(Z^{(n+1)}_{n+1})} = e^{-i\pi\tau(q_{n+1}^2/R_{n+1}^2)}e^{-2\pi i(q_{n+1}^2/R_{n+1}^2)Z_{n+1}}. \tag{74}$$

Comparing the first lines of Eqs. (69), (72), and (74), we see that the condition

$$-\bar{s}_n + s_{n+1} = mq_n, \quad m \text{ integer} \tag{75}$$

must be fulfilled. This reduces the number of allowed products of conformal blocks from $q_nR_{n+1}^2$ to R_{n+1}^2. A further reduction is obtained from requiring invariance under translations along the τ direction. The only combinations that transform correctly are given by

$$\begin{aligned}\mathcal{H}^{(n+1)}_{\bar{s}_{n+1}}(Z^{(1)}_{n+1},\ldots,Z^{(n+1)}_{n+1}) &= \sum_{j=0}^{q_n=1}(-1)^{tj}\mathcal{H}^{(n)}_0(Z_{n+1}+j\tau)\\ &\times\mathcal{G}^{n+1}_{q_ns}(Z_{n+1}+j\tau),\end{aligned} \tag{76}$$

where, again, $\bar{s}_{n+1}=q_ns$ and $s=0,\ldots,q_{n+1}-1$. Inserting Eq. (73) into the above equation, it is straightforward to verify that it is of the form of Eq. (66). We chose $\bar{s}_n=0$ in Eq. (76) in accordance with the sign convention in Eq. (66). Taking another value amounts only to an overall sign change.

We can also compute the quantum numbers of the wave functions [Eq. (68)] under magnetic translations recursively. Under translations with T_1, the conformal blocks at level $n+1$ pick up a phase given by $(-1)^{r(n+1)}e^{2\pi i\nu_{n+1}\bar{s}_{n+1}}$. Thus, K_1 is given by $K_1=[r(n+1)N_s/2+p_{n+1}N\bar{s}_{n+1}]\bmod N_s$. We can also show that $K_2=t(n+1)N_s/2 \bmod N_s$. As in the previous cases, the different fluxes ϕ_μ in Eq. (17) can be incorporated by a proper choice of r and t.

Hence, we found exactly q_{n+1} conformal blocks with the correct quantum numbers. The reader may also note here that the total number of conformal blocks (before imposing the boundary conditions) is not given by the filling factor. Other choices for the charge vectors may decrease this number considerably and simplify computations. This will be discussed in more detail in the next section.

In conclusion, we have constructed the hierarchy wave functions in torus geometry for all filling fractions ν_n that are obtained by successive condensations of quasielectrons. At level n, we find q_n wave functions that are eigenfunctions of the magnetic translation operators T_1 and $T_2^{q_n}$, confirming the expected ground state degeneracy on the torus. Reinstalling the derivatives and antisymmetrizing, we obtain an explicit expression for the wave functions at filling factor ν_n,

$$\Psi_{\nu_n,\bar{s}_n} = \sum_{I_1,\ldots,I_n}(-1)^\sigma\prod_{k=2}^{n}\prod_{i_k\in I_k}\mathcal{D}_{i_k}^{(k-1)}\prod_{\alpha<\beta}\prod_{\substack{i_\alpha\in I_\alpha\\ j_\beta\in I_\beta}}\vartheta_1(z_{i_\alpha j_\beta}|\tau)^{\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\beta)}}\prod_{\substack{\alpha=1\\ i_\alpha<j_\alpha\in I_\alpha}}^{n}\vartheta_1(z_{i_\alpha j_\alpha}|\tau)^{\kappa_\alpha}\mathcal{H}^{(n)}_{\bar{s}_n}(Z^{(1)}_n,\ldots,Z^{(n)}_n)\exp\left(-\frac{1}{2}\sum_k^N y_k^2\right). \tag{77}$$

The sum runs over all possible ways to divide N particles into n sets of size $M_1,\ldots,M_n$ and $(-1)^\sigma$ is the sign picked up by rearranging the radially ordered particles into the sets.

V. ALTERNATIVE CHARGE VECTORS

We have already mentioned that the charge vectors $\mathbf{Q}^{(\alpha)}$ are not uniquely determined by the wave function. To spell out this ambiguity, we first notice from Eq. (62) that the charge vectors enter into the relative part only through the scalar products $\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\beta)}$. If we introduce the vector $c_\alpha = 1/R_\alpha$, the holomorphic current operator takes the form $J(z)=i\Sigma_\alpha c_\alpha \partial\varphi_\alpha(z)$. The constraints on the vertex operators that are imposed by the short-distance behavior of the electrons, the unit $U(1)$ electric charge, and the filling fraction can be expressed as the following conditions on the n-dimensional vectors $\mathbf{Q}^{(\alpha)}$ and $\mathbf{c}$:

$$\mathbf{Q}^{(\alpha)}\cdot\mathbf{Q}^{(\beta)} = K^{\alpha\beta},$$

$$\mathbf{Q}^{(\alpha)}\cdot\mathbf{c} = 1,$$

$$\mathbf{c}\cdot\mathbf{c} = \nu. \tag{78}$$

These scalar products are invariant under simultaneous $O(n)$ transformations on $\mathbf{Q}^{(\alpha)}$ and $\mathbf{c}$. (It should also be clear how to include quasiholes by introducing a new set of vectors, $\mathbf{l}^{(\alpha)}$.) We note, however, that an $O(n)$ transformation will, in general, give irrational radii, implying that the CFT is not rational. This means that the charge sums cannot in any obvious way be reorganized as a finite sum over conformal blocks, as required by the general analysis in Sec. III. Naively, it would seem that no similar restrictions to rational radii would apply when working in the plane, but it is not unlikely that they would emerge from a study of the edge theory for a finite droplet.

We have not tried to find the most general transformation that leaves Eq. (78) invariant and still maintains rational radii of the fields, but only studied a few examples. Note that it is not obvious that preserving Eq. (78) automatically implies that the c.m. part of the wave function remains the same. On the contrary, this amounts to non-trivial mathematical relations between conformal blocks on rational tori. We have numerically verified such a relation explicitly in the case of $\nu=2/5$ and $\nu=3/7$ and have suggested that similar results hold in general. If, on the contrary, these relations turned out not to hold, it would indicate the presence of inequivalent hierarchy states at the same level and with the same filling fraction.

We now present alternative charge vectors for the fractions 2/5, 4/11, and 3/7, in a basis where $\mathbf{c}=(\sqrt{\nu},0,\ldots,0)$. In this basis, a background charge is needed only for the field φ_1 since the correlators for the remaining fields are neutral by construction. As we shall see, the number of conformal blocks is much smaller in this basis, which simplifies calculations. On the other hand, the hierarchy structure is not manifest; e.g., the computation of $\nu=2/5$ blocks is in no simple way related to the computation at $\nu=1/3$.

For $\nu=2/5$, the charge vectors in this basis are given by the following symmetric expressions:

$$\mathbf{Q}^{(1)} = \left(\frac{5}{\sqrt{10}},\frac{1}{\sqrt{2}}\right),$$

$$\mathbf{Q}^{(2)} = \left(\frac{5}{\sqrt{10}},-\frac{1}{\sqrt{2}}\right), \tag{79}$$

where the sets I_1 and I_2 each contain $N/2$ electrons. Since the radii squared are even integers, the double charge-lattice sums reduce to single sums ($p_i'=1$) and the conformal correlator gives a holomorphic basis of the form

$$\psi_{s,s'}(z_i) = \prod_{i<j\in I_1}\vartheta_1(z_{ij}|\tau)^3 \prod_{a<b\in I_2}\vartheta_1(z_{ab}|\tau)^3 \times \prod_{i\in I_1,a\in I_2}\vartheta_1(z_{ia}|\tau)^2 F_s^{10}(5Z)F_r^2(Z_{12}), \tag{80}$$

where $Z=Z_1+Z_2$, $Z_{12}=Z_1-Z_2$, and

$$F_s^{10}(5Z) = \sum_k e^{i\pi\tau 10(k+s/10)^2}e^{2\pi i(k+s/10)5Z},$$

$$F_{s'}^2(Z_{12}) = \sum_k e^{i\pi\tau 2(k+s'/2)^2}e^{2\pi i(k+s'/2)Z_{12}}. \tag{81}$$

The first conformal block depends on the total c.m., while the second contains only relative coordinates. However, it is still not possible to separate the c.m. dependence, as explained below. A linearly independent basis of 20 states is obtained by taking $s=0,\ldots,9$ and $s'=0,1$. This should be compared with the calculation in Sec. IV B, where the original 720 states had to be reduced to 20 by imposing proper boundary conditions.

To compare with our previous calculation, we change the basis for the φ_1 field to

$$\tilde{F}_{s,t}^{10}(5Z) = \sum_k e^{i\pi\tau 10(k+s/5+t/2)^2}e^{2\pi i(k+s/5+t/2)5Z}, \tag{82}$$

where the parameters have the values $s=0,\ldots,4$ and $t=0,1$. In this parametrization, s labels the c.m. coordinate (in fact, this is precisely s' in Sec. IV B), while the boundary conditions are coded in r and the different combinations of t and s'. An analysis along the lines of that given in the previous examples shows that the following linear combinations give eigenstates of $t_{1,i}^{N_s}$ and $t_{2,i}^{N_s}$:

$$\mathcal{H}_s^{(2)}(Z_1,Z_2) = \tilde{F}_{s,0}^{10}(5Z)F_\alpha^2(Z_{12}) + (-1)^\beta\tilde{F}_{s,1}^{10}(5Z)F_{\alpha+1}^2(Z_{12}), \tag{83}$$

where we choose $\alpha=(N_s-\kappa_1+\phi_1/\pi)\mathrm{mod}\,2$ and $\beta=(N_s-\kappa_1+\phi_2/\pi)\mathrm{mod}\,2$. Note that the boundary conditions require a nontrivial combination of different c.m. functions. Thus, there is no simple way to factor out the total c.m. As already mentioned, we have checked that the wave functions thus obtained are numerically equal to those given in Eq. (58). This amounts to a rather complicated identity between sums of products of generalized theta functions.

Finally, we also give explicit expressions for the charge vectors for $\nu=4/11$ and $\nu=3/7$. In the former, we have different numbers of particles in the two sets; thus, the charge vectors look less symmetric,

$$\mathbf{Q}^{(1)} = \left(\sqrt{\frac{11}{4}}, \frac{1}{2} \right), \quad \mathbf{Q}^{(2)} = \left(\sqrt{\frac{11}{4}}, -\frac{3}{2} \right). \tag{84}$$

For 3/7, we find the three charge vectors

$$\mathbf{Q}^{(1)} = \left(\sqrt{\frac{7}{3}}, \frac{2}{\sqrt{6}}, 0 \right),$$

$$\mathbf{Q}^{(2)} = \left(\sqrt{\frac{7}{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}} \right),$$

$$\mathbf{Q}^{(3)} = \left(\sqrt{\frac{7}{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}} \right). \tag{85}$$

Using these, the number of conformal blocks is greatly reduced compared with the earlier representation [Eq. (12)], but unlike the $\nu=2/5$ case, there remain states that must be excluded by applying the boundary conditions.

VI. NUMERICAL TESTS

We have carried out numerical tests of some of the simplest hierarchy wave functions, namely, the $\nu=2/5$ Jain state and the recently discovered $\nu=4/11$ state, both at level 2, and the level three Jain state at $\nu=3/7$. Here, we present results for the 2/5 state and just comment briefly on the others at the end of the section. We compare our wave functions to the ground states obtained by an exact diagonalization of eight and ten particles using an unscreened Coulomb interaction.

As discussed in Sec. III B, the wave functions we have constructed are not unique because of the freedom associated with the derivative operators $\mathcal{D}^{(\alpha)}$. In particular, for the 2/5 state, we have to define the operator $\mathcal{D}^{(1)}$. A simple set of choices is

$$\mathcal{D}_n^{(1)} = t_1^n, \tag{86}$$

and we shall denote the corresponding wave functions by $\psi_{2/5}^{(n)}$. For the simplest choice of $n=1$, we find that the overlap between the exact ground state and $\psi_{2/5}^{(1)}$ is well above 0.9 for all values of L_1 and $N \leqslant 10$. For some values of L_1, this overlap even exceeds 0.99 (see the lowest lying curves in Figs. 1 and 2).

The results are improved further if we take a linear combination of states $\psi_{2/5}^{(n)}$ with different values of n. We find that the overlap with the exact states and the space spanned by $\{\psi_{2/5}^{(n)}\}_{n=1}^{k}$ for different values of k quickly converges to a number very close to 1—in fact, above 0.99 already when two or three states are taken into account—irrespective of the values of L_1 and $N \leqslant 10$. These results for eight and ten electrons are summarized in Figs. 1 and 2. We note that the small variation of the overlap as a function of L_1 very much resembles the results obtained earlier for the Laughlin state on the torus.[26]

We find the fact that we only need to take two or three states into account to get a wave function as good as the Laughlin state as strong evidence for the correctness of our approach, especially in the light of the fact that the Hilbert space in the $\mathbf{K}=0$ sector for ten particles at $\nu=2/5$ has 26 152 dimensions.

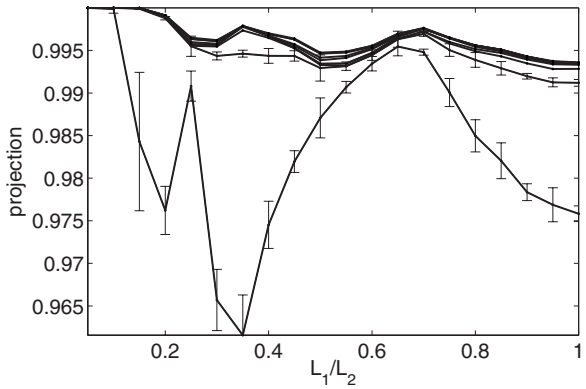

FIG. 1. Projection (amplitude) of the eight-particle exact solution (obtained by diagonalization of an unscreened Coulomb interaction) to the subspace spanned by $\{\psi_{2/5}^{(n)}\}_{n=1}^{k}$, with $k=1,\ldots,9$.

Preliminary results for the state at $\nu=4/11$—one of the recently observed non-Jain states—also seem very promising. However, both the numerical studies and the interpretation thereof are more involved and will be published elsewhere. More over, we have compared the trial wave function at filling fraction $\nu=3/7$ with exact diagonalization results and have found that the simplest choice of the derivative operators [Eq. (24)] already gives a reasonably good description.

VII. SUMMARY AND OUTLOOK

To summarize, we have constructed torus versions of recently proposed wave functions describing the Haldane–Halperin hierarchy of incompressible quantum Hall states in the lowest Landau level. In particular, we managed to incorporate the homogeneous background charge and to derive the nonholomorphic Gaussian factor in a mathematically sound manner. In the previous calculations in the disk geometry, boundary terms had to be ignored,[9] and in a spherical geometry, one typically put a compensating charge at infinity at

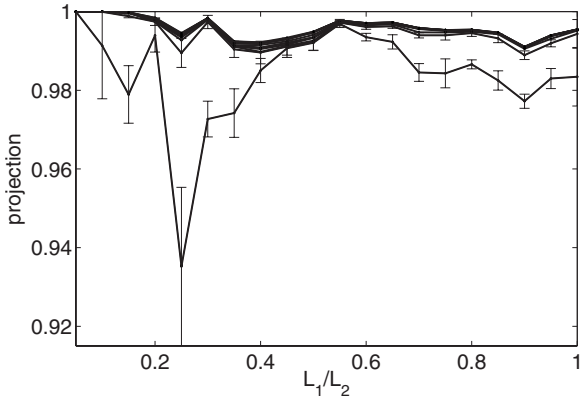

FIG. 2. Projection of the ten-particle exact solution to the subspace spanned by $\{\psi_{2/5}^{(n)}\}_{n=1}^{k}$, with $k=1,\ldots,12$.

the price of not obtaining the Gaussian factor.

At a technical level, we note that all the boson radii R_a are square roots of integers. One can construct wave functions based on CFT where (some of) the R_a's are rational, but these states do not seem to be part of the usual hierarchy and might even be non-Abelian.

We believe that the techniques developed in this paper should also be useful to construct other QH states, for example, the Halperin (n_1,n_2,m) states. It should also be possible to construct quasiparticle states using similar techniques. For the quasiholes in the hierarchy, this should be straightforward since they are described by local vertex operators that are primary fields in the rational CFT's used in our construction. The quasielectrons are more difficult, but a suitable adaptation of the methods developed in Ref. 9 is likely to work. We also believe that the quasielectron excitations of the Moore–Read pfaffian state can be obtained.

Finally, we note that our construction provides a way of investigating the adiabatic continuity from the solvable thin torus case to the bulk for the hierarchy states considered here.

ACKNOWLEDGMENTS

We thank Eddy Ardonne for many helpful discussions and comments on the manuscript. We also thank Fawad Hassan for teaching us about conformal field theory. This work was supported by the Swedish Research Council and NordForsk. J.S. thanks Ari Harju and Risto Nieminen for initiating this collaboration and acknowledges the support of Vilho, Yrjö, and Kalle Väisälä Foundation.

APPENDIX A: THE BACKGROUND CHARGE CONTRIBUTION

In this appendix, we evaluate the integrals in Eqs. (27) and (28) that originate from the homogeneous background charge. We consider a general rectangular integration domain defined by the complex number $\xi=a+ib$

$$\int d^2x \equiv \int_a^{a+L_1} dx \int_b^{b+L_2} dy. \tag{A1}$$

The integral in Eq. (27) is now easily evaluated, and when combined with the second exponential yield factors,

$$\exp\left\{\frac{2\pi i}{R_a}\left[c_a \sum_{\alpha=a+1}^{n} Z_\alpha + q_a Z_a - N_s\left(\frac{\xi}{L_1} - \frac{1+\tau}{2}\right)\right]\right\}. \tag{A2}$$

First, notice that for the choice $\xi=(L_1+iL_2)/2$, corresponding to the symmetric integration region used in the main text, the integral in Eq. (27) vanishes. For an arbitrary ξ, the effect of the integral can be incorporated by shifting the coordinates $z_i \to z_i+\left(\frac{\xi}{L_1}-\frac{1+\tau}{2}\right)$ provided

$$\forall a: \sum_{\alpha=1}^{n} Q_a^{(\alpha)} M_\alpha = C_a N_s. \tag{A3}$$

These relations, which determine the M_α's, were derived previously in the plane as a consequence of the assumption of homogeneity. It is to be noted that exactly the same relations are needed on the torus to guarantee that the c.m. part of the wave function factorizes. Without starting from the plane, we could thus have obtained Eq. (A3) as a consistency condition directly on the torus. Also note that since Eq. (A3) determines the M_α's, it also implies relation (14) that was crucial for obtaining explicit expressions for the c.m. dependence for the hierarchy states. We now turn to the integral in Eq. (28),

$$I_\xi(z,\bar{z}) = \int d^2x' \ln\left|\vartheta_1\left(\frac{z'-z}{L_1}\middle|\tau\right)\right|^2. \tag{A4}$$

Up to a factor independent of z, this integral can be calculated by using the quasiperiodicity [Eq. (37)] of the theta function under lattice translations,

$$\begin{aligned} I_\xi(z) &= \int_a^{a+L_1} dx' \int_b^{b+L_2} dy' \ln\left|\vartheta_1\left(\frac{z'-z}{L_1}\middle|\tau\right)\right|^2 \\ &= \int_{a-x}^{a+L_1-x} dx' \int_{b-y}^{b+L_2-y} dy' \ln\left|\vartheta_1\left(\frac{z'}{L_1}\middle|\tau\right)\right|^2 \\ &= \int_a^{a+L_1} dx' \int_b^{b+L_2} dy' \ln\left|\vartheta_1\left(\frac{z'}{L_1}\middle|\tau\right)\right|^2 \\ &\quad - \int_a^{a+L_1} dx' \int_{b-y}^{b} dy' \left(4\pi\frac{y'}{L_1} + 2\pi \operatorname{Im}\tau\right) \\ &= I_\xi(0) + 2\pi\left\{\left[y-\left(b+\frac{1}{2}L_2\right)\right]^2 \right. \\ &\quad \left. - \left(b+\frac{1}{2}L_2\right)^2\right\}. \end{aligned} \tag{A5}$$

The third identity follows from the fact that the integrand is invariant under lattice translations along the x axis, while under $z \to z+\tau$ it picks up an additional term,

$$\ln\left|e^{-i\pi\tau}e^{-2\pi i z/L_1}\right|^2 = 4\pi y/L_1 + 2\pi \operatorname{Im}\tau. \tag{A6}$$

Again, taking $\xi=-(L_1+iL_2)/2$, we recover Eq. (30) in the text, and taking an arbitrary integration domain, i.e., an arbitrary ξ, just amounts to shifting the coordinate system.

APPENDIX B: DERIVATION OF EQUATION (33)

We assume a compactification radius $R_a^2=2p/p'$, $p,p' \in \mathbb{Z}$—only in this case do the vertex operators (with integer charges) define a rational CFT—and consider the following sum:

$$\begin{aligned} &\sum_{e,m} e^{i\pi\tau\alpha_{e,m}^2} e^{-i\pi\bar{\tau}\bar{\alpha}_{e,m}^2} e^{2\pi i[\alpha_{e,m}Z^{(a)} - \bar{\alpha}_{e,m}\bar{Z}^{(a)}]/R_a} \\ &= \sum_{e,m} e^{i\pi\tau\alpha_{e,m}^2} e^{-i\pi\bar{\tau}\bar{\alpha}_{e,m}^2} e^{(2\pi i/R_a^2)[(e+mR_a^2/2)Z^{(a)} - (e-mR_a^2/2)\bar{Z}^{(a)}]}. \end{aligned} \tag{B1}$$

We write

$$mR_a^2/2 = mp/p' = (\bar{m}p' + r)p/p' = \bar{m}p + rp/p', \quad \text{(B2)}$$

where $r=0,\ldots,p'-1$. Additionally, we write $e+\bar{m}p=2pk_1+s$ and $e-\bar{m}p=2pk_2+s$, with $s=0,\ldots,2p-1$. Thus, we can rewrite the original sum [Eq. (B1)] as

$$\begin{aligned}\sum_{r,s}\sum_{k_1} & e^{i\pi\tau(2pp'k_1+sp'+rp)^2/2pp'} \\ & \times e^{(i\pi/p)(2pp'k_1+sp'+rp)Z^{(a)}} \\ & \times \sum_{k_2} e^{-i\pi\bar{\tau}(2pp'k_2+sp'-rp)^2/2pp'} \\ & \times e^{-(i\pi/p)(2pp'k_2+sp'-rp)\bar{Z}^{(a)}}\end{aligned} \quad \text{(B3)}$$

or, introducing an additional notation, as

$$\sum_{r=0}^{p'-1}\sum_{s=0}^{2p-1} \mathcal{F}_{r,s}(Z^{(a)})\bar{\mathcal{F}}_{-r,s}(\bar{Z}^{(a)}), \quad \text{(B4)}$$

where

$$\mathcal{F}_{r,s}(Z^{(a)}) = \sum_k e^{i\pi\tau(2pp'k+sp'+rp)^2/2pp'} e^{(i\pi/p)(2pp'k+sp'+rp)Z^{(a)}}. \quad \text{(B5)}$$

This is (33); note that Eq. (B4) is a finite sum of factorized terms.

APPENDIX C: THE $\nu=3/7$ AND 4/11 STATES

As a service to the reader that does not want to penetrate the general formalism of Sec. IV, we here provide explicit expressions for the $\nu=3/7$ and 4/11 wave functions. The latter is, as $\nu=2/5$, a level 2 filling fraction, but with a quasielectron density of 1/3. The computation of the torus wave function is in complete analogy to 2/5 when $\mathbf{Q}^{(2)}$ is replaced by $\mathbf{Q}^{(2)}=\left(\frac{2}{\sqrt{3}},\frac{11}{\sqrt{33}}\right)$. The fact that now the sets M_α have different sizes is of no consequence for the calculation. We find $M_1=3M_2\equiv 3M$ for $N=4M$ electrons. Using Eq. (63), but with radius $R_2^2=33$ instead, Eq. (58) now takes the form

$$\mathcal{H}_{\bar{s}}^{(2)}(Z_{4/11}^{(1)},Z_{4/11}^{(2)}) = \sum_{l=0}^{2}(-1)^{tl}\mathcal{G}_{2j}^1(Z_{4/11}^{(1)},Z_{4/11}^{(2)})\mathcal{G}_{11l+3s'}^2(Z_{4/11}^{(2)}). \quad \text{(C1)}$$

The integers r_1, r_2 and t_1, t_2 are fixed by Eq. (17). For $\phi_1=\phi_2=0$, we find $r_1=r_2=N_s-q$ and also $t_1=t_2=N_s-q$. The total wave function can then be written as

$$\begin{aligned}\Psi_{4/11} = & \sum_{\substack{i_1<i_2<\cdots i_{3M}\\ a_1<a_2<\cdots a_M}} (-1)^{(\Sigma_j a_j)}\prod_{k=1}^{M}\mathcal{D}_{a_k}^{(1)}\prod_{i_j<i_l}\vartheta_1(z_{i_j i_l}|\tau)^3 \\ & \times\prod_{a_j<a_l}\vartheta_1(z_{a_j a_l}|\tau)^5\prod_{i_j,a_l}\vartheta_1(z_{i_j a_l}|\tau)^2 \\ & \times\mathcal{H}_{\bar{s}}^{(2)}(Z_{4/11}^{(1)},Z_{4/11}^{(2)})\exp\left(-\frac{1}{2\ell^2}\sum_k y_k^2\right),\end{aligned} \quad \text{(C2)}$$

where

$$Z_{4/11}^{(1)} = 3\sum_{j=1}^{3M} z_{i_j}/L_1 + 2\sum_{j=1}^{M} z_{a_j}/L_1,$$

$$Z_{4/11}^{(2)} = 11\sum_{j=1}^{M} z_{a_j}/L_1. \quad \text{(C3)}$$

As was the case for $\nu=2/5$, the derivatives in Eq. (C2) are not unique, and each choice yields a trial wave function.

The filling fraction $\nu=3/7$ is constructed by three vertex operators with charge vectors $\mathbf{Q}^{(1)}=\left(\frac{3}{\sqrt{3}},0,0\right)$, $\mathbf{Q}^{(2)}=\left(\frac{2}{\sqrt{3}},\frac{5}{\sqrt{15}},0\right)$, and $\mathbf{Q}^{(3)}=\left(\frac{2}{\sqrt{3}},\frac{2}{\sqrt{15}},\frac{7}{\sqrt{35}}\right)$. The sets are of the same size M for $N=3M$ electrons. Again, we set both solenoid fluxes to zero; thus, boundary conditions require $t=r=N_s-3$. Then, Eq. (66) takes the form

$$\begin{aligned}\mathcal{H}_{\bar{s}}^{(3)}(Z_{3/7}^{(1)},Z_{3/7}^{(2)},Z_{3/7}^{(3)}) = & \sum_{l_2=0}^{2}\sum_{l_3=0}^{4}(-1)^{t(l_2+l_3)} \\ & \times\mathcal{G}_{2l_2+2l_3}^1(Z_{3/7}^{(1)})\mathcal{G}_{5l_2+2l_3}^2(Z_{3/7}^{(2)}) \\ & \times\mathcal{G}_{7l_3+5s_3'}^3(Z_{3/7}^{(3)}),\end{aligned} \quad \text{(C4)}$$

with the complete wave function given by

$$\begin{aligned}\Psi_{3/7} = & \sum_{\substack{i_1<i_2<\ldots i_M\\ a_1<a_2<\cdots a_M\\ \alpha_1,\alpha_2\cdots\alpha_M}} (-1)^{\sigma}\prod_{k=1}^{M}\mathcal{D}_{a_k}^{(1)}\prod_{l=1}^{M}\mathcal{D}_{\alpha_l}^{(2)} \\ & \times\Bigg[\prod_{i_j<i_l}\vartheta_1(z_{i_j i_l}|\tau)^3\prod_{a_j<a_l}\vartheta_1(z_{a_j a_l}|\tau)^3\prod_{\alpha_j<\alpha_l}\vartheta_1(z_{\alpha_j\alpha_l}|\tau)^3 \\ & \times\prod_{i_j,a_l}\vartheta_1(z_{i_j a_l}|\tau)^2\prod_{i_j,\alpha_l}\vartheta_1(z_{i_j\alpha_l}|\tau)^2\prod_{a_j,\alpha_l}\vartheta_1(z_{a_j\alpha_l}|\tau)^2\mathcal{H}_{\bar{s}}^{(2)} \\ & \times(Z_{3/7}^{(1)},Z_{3/7}^{(2)},Z_{3/7}^{(3)})\Bigg]\exp\left(-\frac{1}{2\ell^2}\sum_k y_k^2\right).\end{aligned} \quad \text{(C5)}$$

As explained earlier, $(-1)^\sigma$ is the sign picked up by rearranging the ordered fermions into sets. Antisymmetrization requires the derivatives $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ to be different.

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[27] In general, the bosons have unequal compactification radii. Hence, there is, in general, no $O(n)$ symmetry.
[28] An obvious way to preserve the correct symmetry would be to introduce a suitably chosen covariant derivative. This will, however, necessarily introduce $\bar{z}_i$'s in the wave functions, thus reintroducing the need for projection and destroying an appealing feature of our approach. More importantly, in the simplest cases like $\nu=2/5$, it turns out that the wave functions are entirely in the second LL, and thus vanish after projection to the LLL on the torus.
[29] We have, in fact, found that taking $D^{(1)}=t_1$, $D^{(2)}=t_1^2$ gives a wave function identical to the one obtained using Eq. (24). We do not have an analytical understanding of this, but it does suggest the very simple ansatz $\mathcal{D}^{(k)}=t_1^k$.